Flow of an inviscid fluid past a sphere in a pipe

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(Received 19 August 1963)

1. Introduction

The problem of flow of an inviscid, incompressible fluid inside a circular pipe, with a sphere on the axis of the pipe, has been studied by Lamb (1926) (irrotational flow), Long (1953) and Fraenkel (1956) (swirling flow). Because of the difficulty of satisfying all the boundary conditions in the problem, only approximate solutions, valid for spheres of small diameter (compared with that of the pipe) have been obtained. In this paper, it is found that by introducing a vortex sheet over a segment of the diameter of the sphere, flow patterns can be obtained by an inverse method for the case of large spheres. Four different types of flow are considered: (1) irrotational flow, (2) swirling flow with constant axial and angular velocities far upstream, without lee waves, (3) swirling flow with constant axial and angular velocities far upstream, with lee waves, and (4) rotational flow with a paraboloidal velocity distribution far upstream.

2. Governing equations

Cylindrical co-ordinates $(r,\ \theta,\ z)$ will be used. In these co-ordinates, the velocity components will be denoted by u, v and w, respectively. The equations of motion for steady flow are, neglecting viscosity,

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial}{\partial r}\left(\frac{p}{\rho}+\Omega\right), \tag{1}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=0, \tag{2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial}{\partial z}\left(\frac{p}{\rho}+\Omega\right), \tag{3}$$

where z is the axis of symmetry, ρ the (constant) density, and Ω the potential of the external forces (gravity). The equation of continuity for an incompressible fluid is

$$\frac{\partial(ru)}{\partial r}+\frac{\partial(rw)}{\partial z}=0. \tag{4}$$

Equation (4) permits the use of the Stokes stream function ψ, in terms of which the velocity components become

$$u=-\frac{\partial\psi}{r\partial z},\quad w=\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{5}$$

Equation (2) expresses the conservation of angular momentum because it can be written as

$$u\frac{\partial(rv)}{\partial r}+w\frac{\partial(rv)}{\partial z}=0.$$

Consequently rv is a function of ψ alone. For convenience we take

$$(rv)^2=f(\psi). \tag{6}$$

Furthermore, equations (1) and (3) can be written as

$$\frac{\partial\chi}{\partial r}=\frac{v^2}{r}+\frac{1}{2}\frac{\partial v^2}{\partial r}-w\eta, \tag{7}$$

and

$$\frac{\partial\chi}{\partial z}=\frac{1}{2}\frac{\partial v^2}{\partial z}+u\eta, \tag{8}$$

where

$$\chi=\tfrac{1}{2}(u^2+v^2+w^2)+p/\rho+\Omega, \tag{9}$$

and

$$\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}. \tag{10}$$

From equations (7) and (8), together with the continuity equation (4), we have

$$u\frac{\partial\chi}{\partial r}+w\frac{\partial\chi}{\partial z}=0. \tag{11}$$

Therefore, χ also is a function of ψ alone, say

$$\chi=H(\psi). \tag{12}$$

Using equation (5), and noting that

$$\frac{v^2}{r}+\frac{1}{2}\frac{\partial v^2}{\partial r}=\frac{1}{2r^2}\frac{\partial(r^2v^2)}{\partial r}=\frac{1}{2r^2}\frac{\partial f(\psi)}{\partial r},$$

we can write equations (7) and (8) as

$$\frac{\partial H(\psi)}{\partial r}=\frac{1}{2r^2}\frac{\partial f(\psi)}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial r}\eta, \tag{13}$$

$$\frac{\partial H(\psi)}{\partial z}=\frac{1}{2r^2}\frac{\partial f(\psi)}{\partial z}-\frac{1}{r}\frac{\partial\psi}{\partial z}\eta. \tag{14}$$

Multiplying (13) by dr, (14) by dz, and adding, we get

$$dH=\frac{1}{2r^2}df-\frac{\eta}{r}d\psi, \quad \text{or} \quad -\frac{\eta}{r}=\frac{dH}{d\psi}-\frac{1}{2r^2}\frac{df}{d\psi}.$$

Since

$$\eta=-\frac{1}{r}\left(\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}\right),$$

we have, finally,

$$\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}=r^2\frac{dH}{d\psi}-\frac{1}{2}\frac{df}{d\psi}. \tag{15}$$

Equation (15) is the equation governing all the cases mentioned in the introduction. It was derived originally by Long (1953). $H(\psi)$ and $f(\psi)$ are to be determined from the conditions far upstream.

Case 1. The upstream conditions in this case are characterized by

$$w = W(\text{const.}), \quad u = v = 0.$$

Thus $\psi_\infty = \frac{1}{2}r^2 W$ and $H(\psi) = f(\psi) = 0.$

The governing equation is

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = 0. \tag{16}$$

Cases 2 *and* 3. The upstream conditions are characterised by

$$w = W(\text{const.}), \quad u = 0, \quad v/r = \omega(\text{const.}).$$

Thus $\psi_\infty = \frac{1}{2}r^2 W, \quad f(\psi) = r^2v^2 = r^4\omega^2 = \sigma^2\psi^2,$

where σ, the reciprocal of a Rossby number, is $2\omega/W$,

$$H(\psi) = \tfrac{1}{2}(W^2+v^2) + (p_\infty/\rho + \Omega) = \tfrac{1}{2}(W^2+v^2) + \tfrac{1}{2}v^2,$$

and $$\frac{dH}{d\psi} = \frac{d}{d\psi}v^2 = \frac{d}{d\psi}(2\psi\omega^2/W) = \tfrac{1}{2}\sigma^2 W.$$

The governing equation is

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\psi^2}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \sigma^2\psi = \tfrac{1}{2}r^2\sigma^2 W. \tag{17}$$

Case 4. The upstream conditions are $u = v = 0$, and $w = 1-r^2$. Therefore

$$\psi_\infty = \tfrac{1}{2}r^2 - \tfrac{1}{4}r^4, \quad f(\psi) = 0 \quad \text{and} \quad H(\psi) = \tfrac{1}{2}w^2 = \tfrac{1}{2} - 2\psi.$$

The governing equation is

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = -2r^2. \tag{18}$$

The boundary conditions are the same for all the above equations. They are

$$\psi = 0 \quad \text{at} \quad r = 0 \quad \text{and} \quad r^2+z^2 = R^2,$$
$$\psi = \text{const.} \quad \text{at} \quad r = 1.$$

3. Method of solution

With a view to satisfying the boundary conditions at $r = 0$ and $r = 1$, we can write the solutions for the equations (16), (17) and (18) all in the same form:

$$\psi_- = \psi_0 + \sum_{n=1}^{\infty} A_n e^{k_n z} r J_1(\lambda_n r) \quad \text{for} \quad z < 0, \tag{19}$$

$$\psi_+ = \psi_0 + \sum_{n=1}^{\infty} B_n e^{-k_n z} r J_1(\lambda_n r) \quad \text{for} \quad z > 0, \tag{20}$$

In equations (19) and (20), J_1 is the Bessel function of the first order and first kind and λ_n are the roots of $J_1(\lambda) = 0$. For case (1)

$$\psi_0 = \tfrac{1}{2}r^2 W \quad \text{and} \quad k_n = \lambda_n;$$

for case (2) $\psi_0 = \frac{1}{2}r^2 W$ and $k_n = (\lambda_n^2 - \sigma^2)^{\frac{1}{2}}$ $(\sigma < \lambda_1)$;

for case (3) $\psi_0 = \frac{1}{2}r^2 W$ and $k_n = (\lambda_n^2 - \sigma^2)^{\frac{1}{2}}$ $(\lambda_N < \sigma < \lambda_{N+1})$;

and for case (4) $\psi_0 = \frac{1}{2}r^2 - \frac{1}{4}r^4$ and $k_n = \lambda_n$.

In cases (1), (2) and (4), k_n are positive and real; therefore there are no waves. The coefficients A_n and B_n are determined by demanding that

$$\psi_- = \psi_+ \quad \text{at} \quad z = 0, \tag{21}$$

$$\partial\psi_-/\partial z - \partial\psi_+/\partial z = f(r) \quad \text{at} \quad z = 0, \tag{22}$$

in which $f(r) = 0$ for $r \geqslant s$, where s is some positive constant smaller than R (radius of the sphere). Since ψ_- and ψ_+ satisfy the governing equation, (21) and (22) ensure that ψ_+ is the analytic continuation of ψ_-, and there are no singularities in the domain outside of the sphere.

Equation (21) demands that

$$A_n = B_n, \tag{23}$$

and from equation (22), we obtain

$$A_n = \frac{1}{k_n[J_2(\lambda_n)]^2}\int_0^1 f(r)\, J_1(\lambda_n r)\, dr. \tag{24}$$

In case (3), $k_1, k_2, \ldots, k_N$ are imaginary; therefore, there are N wave components in the solution. If we demand that there be no waves upstream, an assumption analogous to that made in the classical theory of surface waves (Rayleigh 1883), then

$$\psi_- = \tfrac{1}{2}Wr^2 + \sum_{N+1}^{\infty} A_n e^{+k_n z} r J_1(\lambda_n r) \quad \text{for} \quad z < 0, \tag{25}$$

$$\psi_+ = \tfrac{1}{2}Wr^2 + \sum_{n=1}^{N} (B_n \cos a_n z + C_n \sin a_n z)\, r J_1(\lambda_n r) + \sum_{N+1}^{\infty} D_n e^{-k_n z} r J_1(\lambda_n r) \quad \text{for} \quad z > 0, \tag{26}$$

where $a_n = (\sigma^2 - \lambda_n^2)^{\frac{1}{2}}$.

Equation (21) demands that

$$A_n = D_n \quad (n > N), \tag{27}$$

$$B_n = 0 \quad (n \leqslant N), \tag{28}$$

and (22) that
$$A_n = \frac{1}{k_n[J_2(\lambda_n)]^2}\int_0^1 f(r)\, J_1(\lambda_n r)\, dr \quad (n > N), \tag{29}$$

$$C_n = -\frac{2}{a_n[J_2(\lambda_n)]^2}\int_0^1 f(r)\, J_1(\lambda_n r)\, dr \quad (n \leqslant N). \tag{30}$$

The C_n are the amplitudes of the N-wave components and the corresponding wave lengths are $2\pi/a_n$. Since there are infinitely many functions $f(r)$ that will give the same C_n's from equation (30), and since different $f(r)$'s generate different shapes of the obstacle, for the same Rossby number, the amplitudes and wave-lengths of the lee waves do not depend on the detailed shape of the obstacle but on certain integrals of the singularity function generating the obstacle. Near the obstacle, however, the flow depends on all the Fourier coefficients of $f(r)$ and is therefore greatly affected by the shape of the obstacle.

It should be remarked that Squire (1955) has pointed out that equation (15) suggests two main possibilities for $\sigma R \to \infty$. The first possibility is that the azimuthal component of vorticity increases with σR, no matter how large the value of σR; this permits the existence of a uniform flow far upstream. The second possibility is that the azimuthal component of vorticity remains bounded when σR becomes sufficiently large; in this case equation (15) shows that the stream function is approximately a function only of radial distance from the axis so that the flow is cylindrical and uniform flow far upstream is impossible. Taylor's experiments indicate that the second alternative is the one found in practice for sufficiently large σR; hence the present solutions apply only for a limited range of σR. The upper limit of this range, where flow with no disturbances far upstream begins to give way to cylindrical flow (the 'Taylor column' phenomenon), is as yet unknown; for a real fluid this upper limit is probably a function of diameter ratio, and of Reynolds number.

4. The general function $f(r)$

It should be pointed out that there is no *a priori* reason for a function (or functions) generating a sphere exactly to exist. What we attempt here is to find a function which will generate an obstacle as close to a sphere as possible.

Two things are considered in the choice of such a function. (1) It should allow the Fourier coefficients to be obtained without much effort. (2) It should allow ample latitude for improving the results. With these considerations in mind, the following is chosen as the generating function

$$\begin{aligned} f(r) &= G_1 + G_2 r^2 + G_3 r^4 \quad \text{for} \quad 0 \leqslant r \leqslant s, \\ &= 0 \qquad\qquad\qquad\quad\; \text{for} \quad r \geqslant s. \end{aligned} \tag{31}$$

Since $G_1 + G_2 s^2 + G_3 s^4 = 0$, only three of the four constants (G_1, G_2, G_3 and s) can be varied independently. It appears therefore, that we can only make three points on the sphere satisfy the condition $\psi = 0$. It turns out however that, if the three points are properly chosen, this particular form of generating function will generate an obstacle whose shape is only slightly different from that of a sphere (see figures 1–4). With the help of a computer, little effort is required in the trial process.

With the generating function given by (31), the coefficients A_n and C_n are given by

$$A_n = 1[G_1 D_1(n) + G_2 D_2(n) + G_3 D_3(n)]/k_n[J_2(\lambda_n)]^2, \tag{32}$$

$$C_n = -2[G_1 D_1(n) + G_2 D_2(n) + G_3 D_3(n)]/a_n[J_2(\lambda_n)]^2, \tag{33}$$

where

$$D_1(n) = -J_0(\lambda_n s)/\lambda_n + 1/\lambda_n, \quad D_2(n) = s^2 J_2(\lambda_n s)/\lambda_n,$$

$$D_3(n) = \frac{2s^4 J_2(\lambda_n s)}{\lambda_n} - \frac{8s^3 J_3(\lambda_n s)}{\lambda_n^2} + \frac{s^4 J_4(\lambda_n s)}{\lambda_n}.$$

Four flow patterns are given at the end of this paper; each represents one of the four cases. Numerical results are given in the next section.

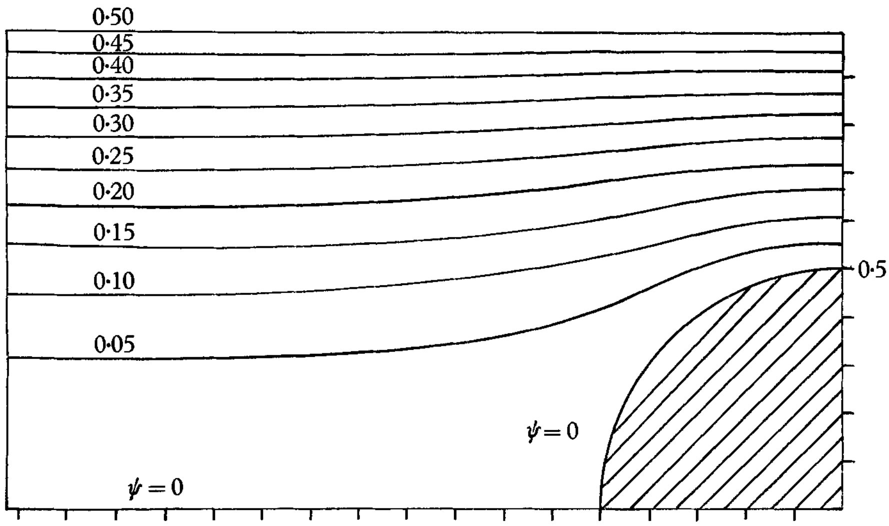

FIGURE 1. Flow pattern for irrotational flow past a sphere inside a pipe of radius 1. Radius of sphere = 0·5.

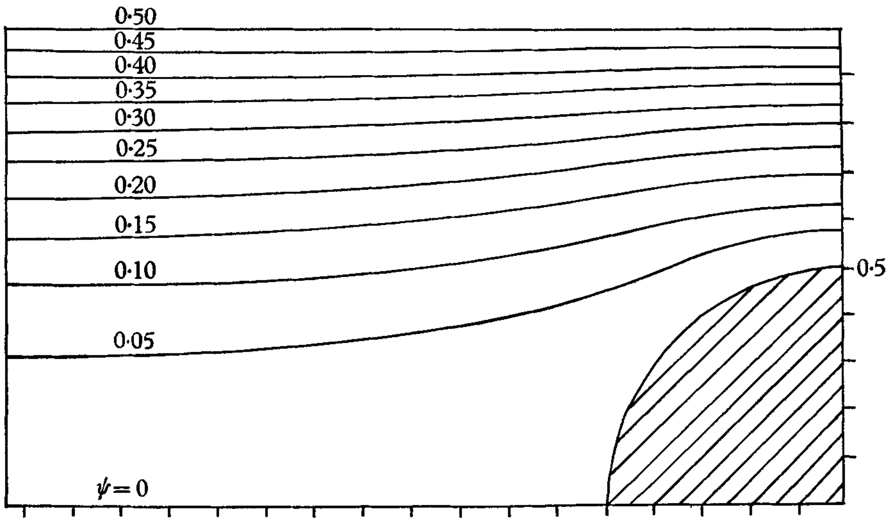

FIGURE 2. Flow pattern for swirling flow past a sphere inside a pipe of radius 1. Rossby number = $\frac{1}{3}$. Radius of sphere = 0·5.

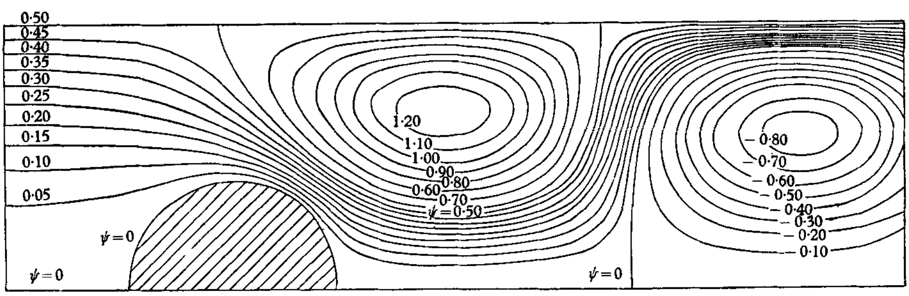

FIGURE 3. Flow pattern for swirling flow past a sphere inside a pipe of radius 1. Rossby number = $\frac{2}{9}$. Radius of sphere = 0·4.

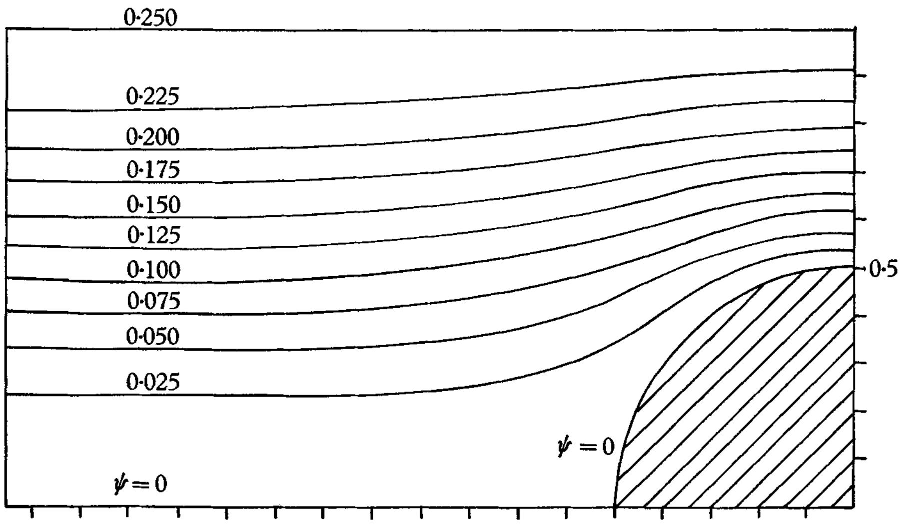

FIGURE 4. Flow pattern for rotational flow past a sphere inside a pipe of radius 1. $w = 1-r^2$ upstream. Radius of sphere = 0·5.

5. Numerical results

For simplicity, W is taken to be unity for cases (1), (2) and (3). For case (4), $w_\infty = 1-r^2$, and the maximum velocity at far upstream is also equal to unity. This amounts to making all the velocities dimensionless through division by $(w_\infty)_{\max.}$.

Figure 1 is for irrotational flow. The ratio of the diameter of the sphere to that of the pipe is 0·5. The generating function is

$$f(r) = 33{\cdot}90(700{\cdot}4r^4+2{\cdot}0r^2-0{\cdot}8) \quad \text{for} \quad 0 \leqslant r \leqslant 0{\cdot}18,$$
$$= 0 \quad \text{for} \quad r \geqslant 0{\cdot}18.$$

Figure 2 is for swirling flow with Rossby number equal to $\frac{1}{3}$, which is larger than the critical number $1/\lambda_1$. There is no wave in the lee of the sphere. The ratio of the diameters is 0·5, and

$$f(r) = 3{\cdot}697(202{\cdot}3r^4+2{\cdot}0r^2-1{\cdot}4) \quad \text{for} \quad 0 \leqslant r \leqslant 0{\cdot}28,$$
$$= 0 \quad \text{for} \quad r \geqslant 0{\cdot}28.$$

Figure 3 is for swirling flow with Rossby number equal to 1/4·5, which is smaller than the critical number. There is one lee-wave component in the lee of the sphere. The ratio of the diameters is 0·4 and

$$f(r) = 13{\cdot}46(137{\cdot}2r^4+2{\cdot}0r^2-1{\cdot}0) \quad \text{for} \quad 0 \leqslant r \leqslant 0{\cdot}28,$$
$$= 0 \quad \text{for} \quad r \geqslant 0{\cdot}28.$$

Figure 4 is for rotational flow with a paraboloidal velocity distribution far upstream. The diameter ratio is 0·5 and

$$f(r) = 100{\cdot}3(7800r^4+2{\cdot}0r^2-0{\cdot}8) \quad \text{for} \quad 0 \leqslant r \leqslant 0{\cdot}10,$$
$$= 0 \quad \text{for} \quad r \geqslant 0{\cdot}10.$$

It should be remarked that the flow pattern given in the eddy region of figure 3 cannot be taken seriously because the flow in that region does not originate at infinity and therefore is not governed by equation (17). In fact, so long as viscosity is taken to be exactly zero, no unique flow can be obtained for that region. It is worth-while to mention that in the case of vanishingly small viscosity Batchelor (1956) has shown that for rotationally symmetric flow, provided the streamlines of the components of velocity in the axial plane are not bounded internally by a solid boundary or a singular surface (which is true in our case), the flow in the 'inviscid core' inside the eddy is governed by the equations $\eta/r = \alpha$ and $rv = \beta$. The constants α and β, however, have to be determined by the need for steadiness in the viscous boundary layer surrounding the eddy. Furthermore, the possible existence of a succession of small eddies of diminishing size at the 90° corners will give rise to some difficulty in the determination of the shape of the singular surface surrounding the inviscid core.

The author wishes to thank Professor C.-S. Yih for his suggestion of the method.

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